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ON THE FLOW FIELD INDUCED BY AN OSCILLATING DISK SUBMERGED IN A SEMI-INFINITE VISCOUS FLUID WITH A SURFACTANT SURFACE FILM

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Abstract—On the basis of the unsteady Stokes' equation the problem of the flow field induced by an oscillating disk fully immersed in a semi-infinite viscous fluid with a surfactant surface layer is solved. The effect of the insoluble surfactant on the hydrodynamics of the oscillating disk is found for varying values of the ratio of the coefficient of the surface shear viscosity to the coefficient of viscosity of the substrate fluid, and depth of the disk below the surface. A new theoretical analysis for obtaining the surface shear viscosity is suggested.

1. INTRODUCTION

The motion of a particle moving in the presence of a fluid–fluid interface is of significant importance and interest in chemical engineering science and applications. It is well known that if the bulk phase of a flow contains substances having great affinity for the fluid surface the composition of the interfacial region is significantly different from that of bulk phase. Recent evidence has indicated that the films forming at crude oil–water interfaces are the result of naturally occurring surface active components in the crude oil rather than oxidation products resulting from exposure of the oil to air.

The rheological properties of the surfactant films at liquid interfaces have been investigated by numerous authors. For insoluble surfactants the effect on the dynamics of the substrate fluid is described by means of the surface shear viscosity. Boussinesq (1913) was the first to propose a two-dimensional analog of the three-dimensional Newtonian fluid to explain the retardation in the terminal velocities of drops and bubbles, the so-called "rising bubble paradox". Scriven (1960) and Slattery (1964) formulated the mathematical equation required to describe the dynamic conditions that exist at an interface with a surface viscosity.

In order to quantitatively assess the role of surface viscosity it is necessary to be able to measure the surface viscosity. The first measurements of surface viscosity apparently were made by Wilson & Ries (1923), who employed a torsional or oscillating disk type of surface viscometer. Langmuir & Schaefer (1937) employed slit and canal viscometers to study insoluble monolayers. Davies & Rideal (1963) presented a concise description of the methods that were developed through 1961. It is interesting to note that none of these methods permit the mathematical analysis which is needed for an exact relationship between surface viscosity and the experimentally measured variables.

Burton & Mannheimer (1965) developed the "deep-channel surface viscometer" which permitted the mathematical analysis necessary to relate surface viscosity to experimentally measurable parameters. Burton & Mannheimer's analysis is restricted only to Newtonian interfaces but Pintor *et al.* (1971) showed that the "deep-channel surface viscometer" can be used to measure the surface viscosity of solutions of macromolecules (polymers).

Goodrich (1969), Goodrich & Chatterjee (1970) and Shail (1978) have examined the dynamics of a rotating disk viscometer. In order to measure the surface viscosity the apparatus consists of a thin disk inserted into the plane interface between the surfactant film and the underlying substrate. The torque necessary to maintain the slow steady rotation is measured and compared with that given in terms of the ratio ϵ/μ by the theoretical formulae derived from the steady Stokes' equations. Here ϵ is the shear surface viscosity and μ the bulk viscosity. The comparison enables evaluation of the surface shear viscosity. Since these methods are insufficiently sensitive for obtaining the small quantities of shear viscosity it is of interest to have an alternative configuration of a completely immersed rotating body in the substrate fluid. This system avoids the evaluation of film torques such as those that act on the edge of the disk.

Recently Shail (1979) has suggested a new theoretical analysis for obtaining of the values of surface shear viscosity.

In the present paper on the basis of the unsteady Stokes' equation we have solved the problem of slow oscillations of the thin circular disk immersed in a semi-infinite fluid whose surface is contaminated with an immiscible surfactant layer. Using the Williams' method (1962) we reduced this problem to the solution of a Fredholm integral equation of the second kind. The reduced integral equation was solved both asymptotically and numerically. Finally the problem of computing surface viscosity from obtained azimuthal velocity and the experimental results is discussed. The numerical results are shown graphically.

2. FORMULATION OF THE PROBLEM

Consider a semi-infinite incompressible viscous fluid on whose plane horizontal surface is a thin layer of immiscible surfactant a few molecules thick. We denote by ϵ the surface shear viscosity of the surfactant film and by ν the kinematic viscosity of the substrate. A thin circular disk fully immersed in the fluid is forced to execute a torsional oscillation about its vertical axis with frequency ω. Let (ρ', φ, z') be cylindrical polar coordinates such that the z'-axis coincides with the axis of the disk, the disk wall lies in the plane $z' = 0$ and z' is measured vertically downwards. It is assumed that the radius of the disk is R and the equation of the plane horizontal interface is $z' = -h'$, where $h' > 0$. We shall suppose that the circular disk oscillates with velocity $U_0 e^{i\omega t}$ and seek a solution that is independent of φ. With R as a typical geometrical length, ω^{-1} as a typical time and $U_0 = \omega R$ as a typical velocity the dimensionless unsteady Stokes' equation and continuity equation have the form:

$$M^2 \frac{\partial \bar{v}}{\partial \tau} = -\nabla p - \operatorname{rot}(\operatorname{rot} \bar{v}) \qquad [1]$$

$$\operatorname{div} \bar{v} = 0 \qquad [2]$$

where $\bar{v} = (v_\rho, v_\varphi, v_z)$ is the velocity, p the dynamic pressure, $\tau = \omega t$ is the dimensionless time and $M^2 = (\omega R^2/\nu)$ the rotational Reynolds number. If one takes into account only the primary flow in the φ direction then from [1] and [2] it follows that the pressure p is constant throughout the fluid and the velocity v_φ satisfied the equation:

$$M^2 \frac{\partial v_\varphi}{\partial \tau} = \frac{\partial^2 v_\varphi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v_\varphi}{\partial \rho} + \frac{\partial^2 v_\varphi}{\partial z^2} - \frac{v_\varphi}{\rho^2} \qquad [3]$$

By setting:

$$v_\varphi(\rho, z, \tau) = v(\rho, z)\, e^{i\tau} \qquad [4]$$

and substituting [4] in [3] the equation [3] reduces to solving the equation:

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{\rho^2} - \beta^2 v = 0 \qquad [5]$$

where $\beta^2 = iM^2$.

The boundary conditions are:

(1) No slip velocity on the disk wall

$$v(\rho, z) = \rho \quad \text{as} \quad 0 \le \rho \le 1,\ z = 0. \tag{6}$$

(2) A balance between the substrate stresses on the adsorbed film and the internal film stresses (Scriven 1960)

$$\frac{\partial v}{\partial z} + \lambda \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v)\right) = 0 \text{ at } z = -h = -\frac{h'}{R} \tag{7}$$

where $\lambda = (\epsilon/\mu R)$ is a dimensionless parameter.

(3) Since the substrate extends to infinity

$$v \to 0 \text{ as } \rho^2 + z^2 \to \infty. \tag{8}$$

From [5], we have

$$\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v)\right) = \beta^2 v - \frac{\partial^2 v}{\partial z^2}. \tag{9}$$

Therefore, from [7], it follows that

$$\frac{\partial v}{\partial z} - \lambda\left(\frac{\partial^2 v}{\partial z^2} - \beta^2 v\right) = 0 \qquad \text{at } z = -h. \tag{7a}$$

It is clear that $\lambda = 0$ corresponds to a clean surface and as $\lambda \to \infty$, [7a] is equivalent to

$$\frac{\partial^2 v}{\partial z^2} - \beta^2 v = 0 \qquad \text{at } z = -h. \tag{10}$$

Thus, [5] yields

$$\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial(\rho v)}{\partial \rho}\right) = 0 \qquad \text{at } z = -h. \tag{11}$$

The general solution of this equation is

$$v(\rho, -h) = A\rho + \frac{B}{\rho}$$

where A and B are arbitrary constants. Since $v(0, -h) = v(\infty, -h) = 0$, one gets $A = B = 0$ or, when $\lambda \to \infty$, $v(\rho, -h) = 0$. Therefore surfactant acts as a rigid plane boundary.

It follows by direct verification that the function

$$w(\rho, \varphi, z) = v(\rho, z)\cos\varphi \tag{12}$$

satisfies [5], [7] and [8] if $v(\rho, z)$ does. So [5] and [7] reduce to the following boundary value problem:

$$(\nabla^2 - \beta^2)w = 0 \tag{13}$$

$$w = \rho \cos \varphi, \qquad z = 0 \tag{14}$$

$$w = 0 \qquad \text{as } \rho^2 + z^2 \to \infty, \tag{15}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}.$$

3. SOLUTION OF THE PROBLEM

The Green's function $G(x, \xi)$ appropriate to the boundary value problem [13]–[15] is

$$(\nabla^2 - \beta^2)G(x, \xi) = -4\pi\delta(x - \xi), \quad x = (\rho, z, \varphi), \quad \xi = (\rho_1, z_1, \varphi_1) \tag{16}$$

$$G(x, \xi) \to 0 \qquad \text{as } \rho^2 + z^2 \to \infty$$
$$\rho_1{}^2 + z_1 \to \infty$$

where $(x, \xi) \in T$ and T is the region in which we seek the solution of the problem [13]–[15]. The integral representation formula for $w(x)$ (Kanwal 1971) is

$$w(x) = -\frac{1}{4\pi}\int_S\int G(x, \xi)\frac{\partial w}{\partial n_1}(\xi)\,\mathrm{d}S' + \frac{1}{4\pi}\int_S\int \rho_1 \cos \varphi_1 \frac{\partial G}{\partial n_1}(x, \xi)\,\mathrm{d}S'$$
$$+ \frac{1}{4\pi}\int_{S_1}\int\left\{G(x, \xi)\frac{\partial w}{\partial z_1}(x) - w(x)\frac{\partial G}{\partial z_1}(x, \xi)\right\}\mathrm{d}S' \tag{17}$$

where S and S_1 are the disk surface and the plane horizontal surface $z = -h$ respectively and $(\partial/\partial n_1)$ denotes differentiation along the outward drawn normal to S. On the surface S_1 we have

$$\lambda \int_0^\infty \left\{G^{(1)}\left(\frac{\partial v}{\partial \rho_1} + \rho_1 \frac{\partial^2 v}{\partial \rho_1{}^2}\right) - v\left(\frac{\partial G^{(1)}}{\partial \rho_1} + \rho_1 \frac{\partial^2 G^{(1)}}{\partial \rho_1{}^2}\right)\right\}\mathrm{d}\rho_1 \tag{18}$$

where $G^{(1)}(\rho, z;\ \rho_1 z_1)$ is the coefficient of $\cos(\varphi - \varphi_1)$ in the Fourier expansion of $G(x, \xi)$. Since $v(\infty, -h) = G^{(1)}(\rho, z;\ \infty, -h) = 0$, $v(0, -h) = 0$ after integrating by parts one obtains that the expression [18] is equal to zero.

In view of Green's theorem we have

$$\int_S\int \rho_1 \cos \varphi_1 \frac{\partial G}{\partial n_1}\,\mathrm{d}S' = \int_S\int G(x, \xi)\frac{\partial}{\partial n_1}(\rho_1 \cos \varphi_1)\,\mathrm{d}S' = 0. \tag{19}$$

Therefore we may represent $w(x)$ as

$$w(x) = \int_S\int \sigma(\xi)G(x, \xi)\cos \varphi_1\,\mathrm{d}S' \tag{20}$$

where

$$\sigma(\xi) = -\frac{1}{4\pi}\rho_1 \frac{\partial}{\partial n_1}\left(\frac{v}{\rho_1}\right).$$

Then the boundary condition on S gives a Fredholm integral equation of the first kind for determining $\sigma(\xi)$:

$$\rho \cos \varphi = \int_S \int \sigma(\xi) G(x, \xi) \cos \varphi_1 \, dS, \qquad x \in S. \tag{21}$$

From the relation [21] it is easy to deduce that

$$\rho = \pi \int_C \sigma(\rho_1, z_1) G^{(1)}(\rho, z; \rho_1, z_1) \rho_1 \, dl_1 \tag{22}$$

where C is the bounding curve of S' in the meridian plane while dl_1 denotes the element of the arc length of C.

For a thin circular disk we have

$$\rho = \pi \int_0^1 \sigma^*(\rho_1, 0) G^{(1)}(\rho, 0; \rho_1 0) \rho_1 \, d\rho_1, \qquad 0 \le \rho \le 1 \tag{23}$$

$$\sigma^* = 2\sigma.$$

The tangential stress component $\bar{\tau}$ on the surface S, in the direction of φ increasing, is

$$\bar{\tau} = e^{i\tau} \rho \frac{\partial}{\partial n} \left(\frac{v}{\rho} \right).$$

Furthermore, considerations of potential theory show that the source density $\sigma(\rho, z)$ on S is related to v by

$$\bar{\tau} = e^{i\tau} \rho \frac{\partial}{\partial n} \left(\frac{v}{\rho} \right) = -4\pi \, e^{i\tau} \sigma(\rho, z). \tag{24}$$

The physical quantity of interest is the frictional torque which is given by

$$N = e^{i\tau} \int_S \int \rho^2 \frac{\partial}{\partial n} \left(\frac{v}{\rho} \right) dS = -8\pi^2 \, e^{i\tau} \int_C \rho^2 \sigma(\rho, z) \, dl.$$

Therefore, for the disk of radius 1:

$$N = -8\pi^2 \, e^{i\tau} \int_0^1 \rho^2 \sigma^*(\rho, 0) \, d\rho. \tag{25}$$

Since the fundamental solution of [16] is $(\exp(-\beta r)/r)$ we shall have

$$G(x, \xi) = \frac{\exp(-\beta r)}{r} + G_1(x, \xi)$$

where

$$(\nabla^2 - \beta^2) G_1(x, \xi) = 0 \tag{26}$$

and $G_1(x, \xi)$ satisfies the boundary condition [7a]

$$\left. \frac{\partial G}{\partial z_1} \right|_{z_1 = -h} - \lambda \left. \left(\frac{\partial^2 G}{\partial z_1^2} - \beta^2 G \right) \right|_{z_1 = -h} = 0.$$

Here

$$r = \sqrt{(\rho^2 + \rho_1{}^2 - 2\rho\rho_1 \cos(\varphi - \varphi_1) + (z - z_1)^2)}.$$

According to Watson (1966) we have

$$\frac{\exp(-\beta r)}{r} = \int_0^\infty \frac{J_0(p\sqrt{(\rho^2 + \rho_1{}^2 - 2\rho\rho_1 \cos\varphi)})}{\sqrt{(p^2+\beta^2)}} \mathrm{e}^{-|z-z_1|\sqrt{(p^2+\beta^2)}} p \,\mathrm{d}p.$$

Here

$$J_0(p\sqrt{\rho^2 + \rho_1{}^2 - 2p p_1 \cos\varphi})) = \sum_{j=0}^{\infty} (2 - \delta_{0j}) \cos j(\varphi - \varphi_1) J_j(p\rho) J_j(p\rho_1)$$

where J_j are Bessel functions of the first kind and δ_{0j} is the Kronecker delta.

Thus

$$2\int_0^\infty p \frac{J_1(p\rho)J_1(p\rho_1)}{\sqrt{(p^2+\beta^2)}} \mathrm{e}^{-|z-z_1|\sqrt{(p^2+\beta^2)}} \mathrm{d}p$$

is the coefficient of $\cos(\varphi - \varphi_1)$ in the Fourier expansion of the fundamental solution of [16].

The next step is to find the coefficient of $\cos(\varphi - \varphi_1)$ in the Fourier expansion of $G_1(x, \xi)$. Applying the method of images the required additional coefficient is

$$2\int_0^\infty p \frac{J_1(p\rho)J_1(p\rho_1)}{\sqrt{(p^2+\beta^2)}} \frac{\sqrt{(p^2+\beta^2)} - \lambda p^2}{\sqrt{(p^2+\beta^2)} + \lambda p^2} \mathrm{e}^{-(z+z_1+2h)\sqrt{(p^2+\beta^2)}} \mathrm{d}p.$$

Therefore

$$G^{(1)}(\rho, z; \rho_1, z_1) = 2\int_0^\infty p \frac{J_1(p\rho)J_1(p\rho_1)}{\sqrt{(p^2+\beta^2)}} \left[\mathrm{e}^{-|z-z_1|\sqrt{(p^2+\beta^2)}} + \frac{\sqrt{(p^2+\beta^2)} - \lambda p^2}{\sqrt{(p^2+\beta^2)} + \lambda p^2} \mathrm{e}^{-(z+z_1+2h)\sqrt{(p^2+\beta^2)}} \right] \mathrm{d}p.$$

Thus, [23] yields

$$\rho = \pi \int_0^1 \sigma^*(\rho_1, 0) 2 \left[\int_0^\infty p \frac{J_1(p\rho)J_1(p\rho_1)}{\sqrt{(p^2+\beta^2)}} \left(1 + \frac{\sqrt{p^2+\beta^2) - \lambda p^2}}{\sqrt{(p^2+\beta^2)} + \lambda p^2} \mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}} \right) \mathrm{d}p \right] \rho_1 \,\mathrm{d}\rho_1 \quad [27]$$

where $0 \leq \rho \leq 1$.

Imposition of the no-slip boundary condition on the oscillating disk leads to a Fredholm integral equation of the first kind whose solution has an integrable singularity at the disk edge. Further, using Williams' method we shall reduce [27] into a Fredholm integral equation of the second kind for a derived function which is regular.

By setting $f(\rho) = 2\pi\rho\sigma^*(\rho, 0)$ we can write [27] as

$$\int_0^1 f(\rho_1) K_0(\rho, \rho_1) \,\mathrm{d}\rho_1 = \rho - \int_0^1 f(\rho_1) \left\{ \int_0^\infty \left[p \left(1 + \frac{\sqrt{(p^2+\beta^2)} - \lambda p^2}{\sqrt{(p^2+\beta^2)} + \lambda p^2} \mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}} \right) \frac{1}{\sqrt{(p^2+\beta^2)}} - 1 \right] J_1(p\rho) J_1(p\rho_1) \,\mathrm{d}p \right\} \mathrm{d}\rho_1 \quad [28]$$

where

$$K_0(\rho, \rho_1) = \int_0^\infty J_1(p\rho)J_1(p\rho_1)\,\mathrm{d}p.$$

Let

$$S(\rho) = \rho \int_\rho^1 \frac{\rho_1^{-1} f(\rho_1)}{\sqrt{(\rho_1^2 - \rho_2)}}\,\mathrm{d}\rho_1.$$

Then the integral equation [28] becomes

$$S(\rho) + \int_0^1 L(\rho, \rho_1) S(\rho_1)\,\mathrm{d}\rho_1 = 2\rho \qquad [29]$$

which is a Fredholm integral equation of the second kind in $S(\rho)$. Here $0 \le \rho \le 1$ and

$$L(\rho, \rho_1) = \frac{2}{\pi}\int_0^\infty \left[p\left(1 + \frac{\sqrt{(p^2+\beta^2)} - \lambda p^2}{\sqrt{(p^2+\beta^2)} + \lambda p^2}\,\mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}}\right)\frac{1}{\sqrt{(p^2+\beta^2)}} - 1\right]\sin p\rho \sin p\rho_1\,\mathrm{d}p.$$

Therefore

$$f(\rho_1) = -\frac{2\rho_1}{\pi}\frac{\mathrm{d}}{\mathrm{d}\rho_1}\int_{\rho_1}^1 \frac{S(\rho)}{\sqrt{(\rho^2 - \rho_1^2)}}\,\mathrm{d}\rho$$

and

$$N = 8\,\mathrm{e}^{i\tau}\int_0^1 \rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\int_\rho^1 \frac{S(x)}{\sqrt{(x^2-\rho^2)}}\,\mathrm{d}x\right\}\mathrm{d}\rho = -16\,\mathrm{e}^{i\tau}\int_0^1 xS(x)\,\mathrm{d}x.$$

The kernel $L(\rho, \rho_1)$ can be written as

$$L(\rho, \rho_1) = \frac{1}{\pi}\{Q(|\rho - \rho_1|) - Q(\rho + \rho_1)\}$$

where

$$Q(v) = \int_0^\infty p\frac{\sqrt{(p^2+\beta^2)} - \lambda p^2}{\sqrt{(p^2+\beta^2)} + \lambda p^2}\frac{\mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}}}{\sqrt{(p^2+\beta^2)}}\cos vp\,\mathrm{d}p$$

$$+ \int_0^\infty \frac{p - \sqrt{(\beta^2+p^2)}}{\sqrt{(p^2+\beta_2)}}\cos vp\,\mathrm{d}p = I_1' + I_2,$$

$$I_2 = \int_0^\infty \frac{p - \sqrt{(\beta^2+p^2)}}{\sqrt{(p^2+\beta^2)}}\cos vp\,\mathrm{d}p = \int_0^\infty \frac{-\beta^2 \cos vp\,\mathrm{d}p}{\sqrt{(p^2+\beta^2)}(p+\sqrt{(p^2+\beta^2)})} = -\beta\frac{i\pi}{2}J_1(i\beta v)$$

$$I_1' = \int_0^\infty p\frac{\sqrt{(p^2+\beta^2)} - \lambda p^2}{\sqrt{(p^2+\beta^2)} + \lambda p^2}\frac{\mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}}}{\sqrt{(\beta^2+p^2)}}\cos vp\,\mathrm{d}p = -\int_0^\infty \frac{p\,\mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}}}{\sqrt{(\beta^2+p^2)}}\cos vp\,\mathrm{d}p$$

$$+2\int_0^\infty \frac{p\,\mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}}}{\sqrt{(p^2+\beta^2)} + \lambda p^2}\cos vp\,\mathrm{d}p = I_1 + 2I_1^\lambda$$

$$I_1 = -\frac{1}{2h}\int_0^\infty \cos vp\,\mathrm{d}\,\mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}} = -\frac{1}{2h}\mathrm{e}^{-2h\beta} - \frac{v}{2h}\int_0^\infty \mathrm{e}^{-2h\sqrt{(p^2+\beta^2)}}\sin vp\,\mathrm{d}p.$$

Therefore

$$Q(v) = -I_1 + I_2 + 2I_1^{\lambda}. \qquad [30]$$

4. NUMERICAL AND ASYMPTOTIC RESULTS

When $h \gg 1$ and $Mh = 0(1)$ we set $\beta h = c$ and $hp = t$. Then

$$I_3 = \int_0^\infty e^{-2h\sqrt{(p^2+\beta^2)}} \sin vp \, dp = \int_0^\infty e^{-2\sqrt{(t^2+c^2)}} \left[\frac{tv}{h} - \frac{t^3v^3}{6h^3} + 0(h^{-5})\right] \frac{1}{h} \, dt$$

$$= \frac{v}{4h^2}(2c+1)\,e^{-2c} - \frac{v^3}{48h^4}(4c^2+6c+3)\,e^{-2c} + 0(h^{-6})$$

$$i\beta J_1(i\beta v) = -\frac{c^2v}{2h^2} - \frac{c^4v^3}{16h^4} - \frac{c^6v^5}{h^6 192} + 0(h^{-8}).$$

There are three cases to be considered:

Case (*i*). The surfactant film has small surface shear viscosity compared with the viscosity of the bulk fluid, i.e. $\lambda = 0(h^{-1})$.

Case (*ii*). The shear surface viscosity ϵ and the viscosity of the bulk fluid μ have the same order of magnitude, i.e. $\lambda = 0(1)$.

Case (*iii*). The film is very viscous compared with the bulk fluid, i.e. $\lambda = 0(h)$ 4(a) Let $\lambda = 0(h^{-1})$ and $\lambda h = r$. Then

$$I_1^{\lambda} = \int_0^\infty \frac{p\,e^{-2h\sqrt{(p^2+\beta^2)}}}{\sqrt{(p^2+\beta^2)}+\lambda p^2} \cos vp \, dp = \frac{1}{h}\int_0^\infty \frac{t\,e^{-2\sqrt{(t^2+c^2)}}}{\sqrt{(t^2+c^2)}} \left(1 - \frac{rt^2}{h^2\sqrt{(t^2+c^2)}} + \frac{r^2t^4}{h^4(t^2+c^2)}\right.$$

$$\left. - \frac{v^2t^2}{2h^2} + \frac{v^4t^4}{24h^4} + \frac{rt^4v^2}{2h^4\sqrt{(t^2+c^2)}} + 0(h^{-6})\right) dt = \text{const} - \frac{v^2}{2h^3}\frac{1}{4}(2c+1)\,e^{-2c}$$

$$+ \frac{v^4}{24h^5}(4c^2+6c+3)\,e^{-2c} + \frac{rv^2}{2h^5}\left[\frac{1}{8}(-4c^3+2c^2+6c+3) + c^4E_1(2c)\right] + 0(h^{-7}).$$

4(b) Let $\lambda = 0(1)$. Then

$$I_1^{\lambda} = \int_0^\infty \frac{p\,e^{-2h\sqrt{(p^2+\beta^2)}}}{\sqrt{(p^2+\beta^2)}+\lambda p^2} \cos vp \, dp = \int_0^\infty \frac{t}{h}\frac{e^{-2\sqrt{(t^2+c^2)}}}{\sqrt{(t^2+c^2)}} \left(\text{const} - \frac{v^2t^2}{2h^2} + \frac{\lambda v^2t^4}{2h^3\sqrt{(t^2+c^2)}}\right.$$

$$\left. + \frac{v^4t^4}{24h^4} - \frac{\lambda^2t^6v^2}{2h^4(t^2+c^2)} + \frac{\lambda^3t^8v^2}{2h^5(\sqrt{(t^2+c^2)})^3} + 0(h^{-6})\right) dt$$

$$= \text{const} - \frac{v^2}{8h^3}(2c+1)\,e^{-2c} + \frac{\lambda v^2}{2h^4}\left[\frac{1}{8}(-4c^3+2c^2+6c+3) + c^4E_1(2c)\right] + 0(h^{-5}).$$

4(c) Let $\lambda = 0(h)$ and $(\lambda/h) = r$. Then

$$I_1^{\lambda} = \int_0^\infty \frac{t}{h}\frac{e^{-2\sqrt{(t^2+c^2)}}}{\sqrt{(t^2+c^2)}+rt^2}\left(1 - \frac{v^2t^2}{2h^2} + \frac{v^4t^4}{24h^4} + 0(h^{-6})\right) dt = \text{const} - A_3\frac{v^2}{2h^3} + \frac{v^4}{24h^5}A_5 + 0(h^{-7})$$

where

$$A_3 = \int_0^\infty \frac{t^3\,e^{-2\sqrt{(t^2+c^2)}}}{\sqrt{(t^2+c^2)}+rt^2}\,dt; \qquad A_5 = \int_0^\infty \frac{t^5\,e^{-2\sqrt{(t^2+c^2)}}}{\sqrt{(t^2+c^2)}+rt^2}\,dt, \quad E_1(2c) = \int_c^\infty \frac{e^{-2q}}{q}\,dq.$$

The asymptotic expansion of $L(v, w)$ in these three cases is:

4(a) $\lambda = 0(h^{-1})$

$$L(v, w) = \frac{1}{\pi h}\begin{cases} -\dfrac{\pi c^2 w}{2h} - \dfrac{\pi c^4 w(3v^2+w^2)}{16h^3} - \dfrac{c^6 w(5v^4+10v^2w^2+w^4)\pi}{192h^5} + \dfrac{vw}{2h^2}(2c+1)\,e^{-2c} \\ -\dfrac{5vw(v^2+w^2)}{12h^4}(4c^2+6c+3)\,e^{-2c} - \dfrac{8vwr}{2h^4}\left[\dfrac{1}{8}(-4c^3+2c^2+6c+3)+c^4E_1(2c)\right] \\ \qquad + 0(h^{-6}), v \geqslant w \\ -\dfrac{\pi c^2 v}{2h} - \dfrac{\pi c^4 v(3w^2+v^2)}{16h^3} - \dfrac{c^2 v(5w^4+10w^2v^2+v^4)\pi}{192h^5} + \dfrac{vw}{2h^2}(2c+1)\,e^{-2c} \\ -\dfrac{5vw(v^2+w^2)}{12h^4}(4c^2+6c+3)\,e^{-2c} - \dfrac{8vwr}{2h^4}\left[\dfrac{1}{8}(-4c^3+2c^2+6c+3)+c^4E_1(2c)\right] \\ \qquad + 0(h^{-6}),\ v < w \end{cases}$$

4(b) $\lambda = 0(1)$

$$L(v, w) = \left\{-\frac{\pi c^2 w}{2h} - \frac{\pi c^4 w(3v^2+w^2)}{16h^3} + \frac{vw}{2h^2}(2c+1)\,e^{-2c} - \frac{8vw}{2h^3}\right.$$

$$\left.\times\left[\frac{1}{8}(-4c^3+2c^2+6c+3)+c^4E_1(2c)\right] + 0(h^{-4})\right\}\frac{1}{\pi h} \text{ as } v \geq w$$

$$L(v, w) = \left\{-\frac{\pi c^2 v}{2h} - \frac{\pi c^4 v(3w^2+v^2)}{16h^3} + \frac{vw}{2h}(2c+1)\,e^{-2c} - \frac{8\lambda vw}{2h^3}\right.$$

$$\left.\times\left[\frac{1}{8}(-4c^3+2c^2+6c+3)+c^4E_1(2c)\right] + 0(h^{-4})\right\}\frac{1}{\pi h} \text{ as } v < w.$$

4(c) $\lambda = 0(h)$

$$L(v, w) = \frac{1}{\pi h}\begin{cases} -\dfrac{\pi c^2 w}{2h} - \dfrac{\pi c^2 w(3v^2+w^2)}{16h^3} - \dfrac{c^6 w(5v^4+10v^2w^2+w^4)\pi}{192h^5} - \dfrac{vw}{2h^2}(2c+1)\,e^{-2c} \\ +\dfrac{vw(v^2+w^2)}{12h^4} + \dfrac{8vw}{2h^2}A_3 - \dfrac{16vw(v^2+w^2)}{24h^4}A_5 + 0(h^{-6}),\ v \geq w \\ -\dfrac{\pi c^2 v}{2h} - \dfrac{\pi c^4 v(3w^2+v^2)}{16h^3} + \dfrac{c^6 v(5w^4+10v^2w^2+v^4)\pi}{192h^5} - \dfrac{vw}{2h^2}(2c+1)\,e^{-2c} \\ +\dfrac{vw(v^2+w^2)}{12h^7} - \dfrac{8vw}{2h^2}A_3 - \dfrac{16vw(v^2+w^2)}{24h^4}A_5 + 0(h^{-6}),\ v \leq w. \end{cases}$$

It was found that

$$N = -16\,e^{i\tau}\int_0^1 xS(x)\,dx. \qquad [31]$$

The Neumann iterative solutions of [29] and the torques in the above three cases are

4(a) The case $\lambda = 0(h^{-1})$

$$S(v) = 2v - \frac{2}{\pi h}\left[-\frac{\pi}{2}\frac{c^2}{h}\left(\frac{v}{2}-\frac{v^3}{6}\right) - \frac{\pi c^4}{16h^3}\left(\frac{3v}{4}+\frac{v^3}{2}-\frac{1}{20}v^5+v^6\right) - \frac{\pi c^6 v}{192h^5}\right.$$

$$\times\left(\frac{5}{6}+\frac{5}{2}v^2+\frac{v^4}{2}-\frac{1}{42}v^6\right)+\frac{(2c+1)\,e^{-2c}}{6h^2}\,v - \frac{5(4c^2+6c+3)\,e^{-2c}}{12h^2}\left(\frac{v^3}{3}+\frac{v}{5}\right)$$

$$\left. -\frac{vr}{6h^4}[(-4c^3+2c^2+6c+3)\,e^{-2c}+8c^4E_1(2c)]+0(h^{-6})\right]$$

$$N = \left\{ -16\left\{\frac{2}{3}-\frac{2}{\pi h}\left[-\frac{\pi c^2}{15h}-\frac{\pi c^4}{16h^3}\frac{201}{280}-\frac{\pi c^6}{378h^5}\frac{1}{3}+\frac{(2c+1)\,e^{-2c}}{18h^2}-\frac{1}{18}\frac{(4c^2+6c+3)\,e^{-2c}}{h^4}\right.\right.\right.$$

$$\left.\left.\left. -\frac{r}{18h^4}[(-4c^3+2c^2+6c+3)\,e^{-2c}+8c^4E_1(2c)]\right]+0(h^{-7})\right\}\right\} e^{i\tau}.$$

4(b) The case $\lambda = 0(1)$.

$$S(v) = 2v - \frac{2}{\pi h}\left[-\frac{\pi c^2}{2h}\left(\frac{v}{2}-\frac{v^3}{6}\right) - \frac{\pi c^4}{16h^3}\left(\frac{3}{4}v+\frac{v^3}{2}-\frac{1}{20}v^5+v^6\right)\right.$$

$$\left. +\frac{(2c+1)\,e^{-2c}}{6h^2}\,v - \frac{v\lambda}{6h^3}[(-4c^3+2c^2+6c+3)\,e^{-2c}+8c^4E_1(2c)]+0(h^{-4})\right]$$

$$N = \left\{ -16\left\{\frac{2}{3}-\frac{2}{\pi h}\left[-\frac{\pi c^2}{15h}-\frac{\pi c^4}{16h^3}\frac{201}{280}+\frac{(2c+1)\,e^{-2c}}{6h^2}-\frac{\lambda}{18h^3}\right.\right.\right.$$

$$\left.\left.\left.\times[(-4c^3+2c^3+6c+3)\,e^{-2c}+8c^4E_1(2c)]\right]+0(h^{-5})\right\}\right\} e^{i\tau}.$$

4(c) The case

$$S(v) = 2v - \frac{2}{\pi h}\left[-\frac{\pi c^2}{2h}\left(\frac{v}{2}-\frac{v^3}{6}\right) - \frac{\pi c^4}{16h^3}\left(\frac{3}{4}v+\frac{v^3}{2}-\frac{1}{20}v^5+v^6\right)\right.$$

$$-\frac{\pi c^6 v}{192h^5}\left(\frac{5}{6}+\frac{5}{2}v^2+\frac{v^4}{2}-\frac{1}{42}v^6\right)-\frac{v}{6h^2}(2c+1)\,e^{-2c}+\frac{1}{12h^4}\left(\frac{v^3}{3}+\frac{v}{5}\right)$$

$$\left. +\frac{8v}{6h^2}A_3-\frac{16}{24h^4}\left(\frac{v^3}{3}+\frac{v}{5}\right)A_5+0(h^{-6})\right]$$

$$N = \left\{ -16\left\{\frac{2}{3}-\frac{2}{\pi h}\left[-\frac{\pi c^2}{h}\frac{1}{15}-\frac{\pi c^4}{16h^3}\frac{201}{280}-\frac{\pi c^6}{378h^5}\frac{1}{3}-\frac{(2c+1)\,e^{-2c}}{18h^2}\right.\right.\right.$$

$$\left.\left.\left. +\frac{1}{30h^2}+\frac{4}{9h^2}A_3-\frac{4}{45h^4}A_5\right]+0(h^{-7})\right\}\right\} e^{i\tau}.$$

In the case of arbitrary h and λ the integral equation [29] is solved numerically by using Gregory's integration formula (Krilov *et al.* 1976). An iterative procedure is organized at which

Table 1.

								h=0.25
λ	0.	0.025	0.05	0.075	0.I	0.25	0.5	0.75
Re	8.I34	8.295	8.444	8.585	8.7I8	9.385	I0.I79	I0.744
I_m	0.848	0.88I	0.9I2	0.943	0.972	I.I27	I.333	I.495
$\tau=\frac{\pi}{4}$	5.I52	5.242	5.327	5.404	5.478	5.839	6.255	6.540

λ	I.	I.25	I.5	I.75	2.	5	50	∞
Re	II.I70	II.505	II.775	II.997	I2.I87	I3.260	I4.287	I4.507
I_m	I.626	I.734	I.825	I.904	I.972	2.396	2.88I	2.993
$\tau=\frac{\pi}{4}$	6.749	6.909	7.035	7.I38	7.223	7.682	8.065	8.I42

Table 2.

					h=0.I25
λ	0.00	0.25	0.5	0.75	I
Re	7.028	8.843	I0.07	I0.99	II.7I2
I_m	0.904	I.453	I.923	2.334	2.694
$\tau=\frac{\pi}{4}$	4.330	5.226	5.76I	6.I2I	6.377

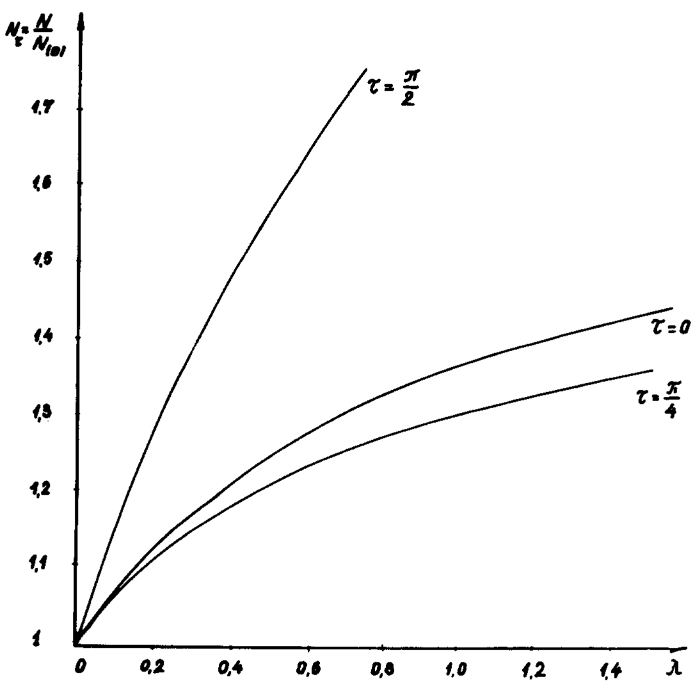

Figure 1. Effect of λ on $N/N(0)$ for various values of τ at $h = 0.25$.

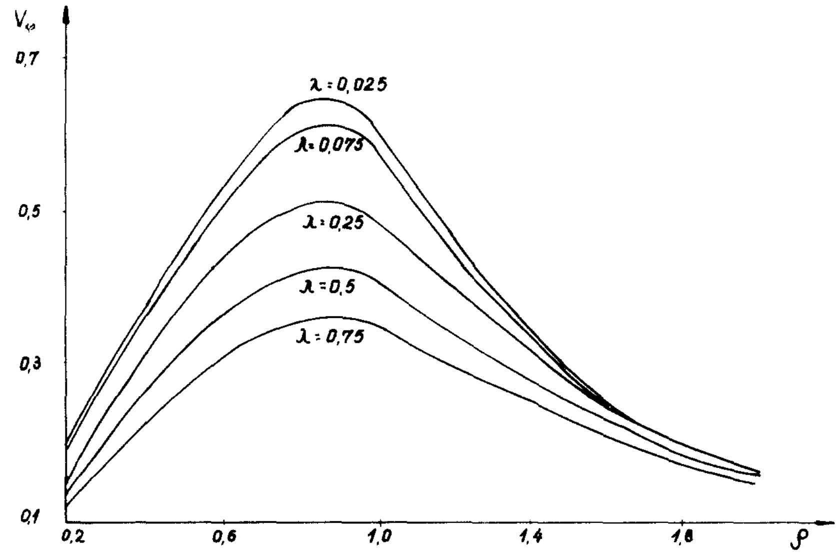

Figure 2. Velocity profiles for values of λ at $h = 0.25$ and $\tau = 0$.

a first approximation to the solution is found by neglecting the difference correction. After calculating $S(\rho)$ at the n pivotal points the torque N is computed by Simpson's rule. Detailed calculations are made at $h = 0.125$, $h = 0.25$, $\tau = 0, (\pi/4), (\pi/2)$, $M = 0.45$ and different values of λ. The results are shown in Tables 1 and 2 and Figure 1. We observe that the increase in the torque N is more rapid for surfactants with small λ. Since the measurement of the azimutal velocity of surface particles was proved to be a better experimental method for determining surface viscosities than torque measurements the surface velocity for varying h and λ is computed.

The function $v(\rho, -h)$ can be found by using the formula

$$v(\rho, -h) = \frac{4}{\pi}\int_0^1 S(x)\left(\int_0^\infty \frac{e^{-h\sqrt{(p^2+\beta^2)}}}{\sqrt{(p^2+\beta^2)}+\lambda p^2}\, pJ_1(p\rho)\sin px\, dp\right) dx. \qquad [32]$$

Then the azimutal surface velocity v_φ is given by

$$v_\varphi(\rho, -h) = \cos\tau \text{ real } v(\rho, -h) - \sin \tau I_m v(\rho, -h). \qquad [33]$$

Since we know the values of the function $S(\rho)$ at the pivotal points it is easy to compute the surface velocity profiles. Figure 2 presents the variation of the surface velocity $v_\varphi(\rho, -h)$ with ρ as λ is varied.

5. CONCLUSION

The paper deals with the flow field induced by an oscillating disk fully immersed in a semi-infinite viscous fluid with a surfactant surface layer. The problem was solved on the basis of the unsteady Stokes' equation for the azimutal velocity.

The boundary condition on the surfactant film includes the ratio λ, of the surface shear viscosity ϵ to the bulk viscosity μ, of the fluid. Imposition of the no-slip boundary condition on the oscillating disk leads to a Fredholm integral equation of the first kind whose solution has an integrable singularity at the disk edge.

Using the Williams' method we reduced the problem to the solution of a Fredholm integral equation of second kind for a derived quantity $S(\rho)$ which is regular on [0, 1]. The last equation is solved both asymptotically and numerically, and the resistive torque on the oscillating disk and surface velocity profiles are computed. The effect of the insoluble surfactant on the hydrodynamics of the oscillating disk is investigated for varying values of the ratio λ and the depth h, of the disk below the surfactant film.

An acceptable procedure for determining the surface shear viscosity ϵ, would be to measure the azimutal velocity v_φ of the suitably marked fluid particle in the surfactant fluid surface and to use the formulas [32] and [33].

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